

# NEW LOWER BOUNDS FOR THE LEAST COMMON MULTIPLES OF ARITHMETIC PROGRESSIONS

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**ABSTRACT.** For relatively prime positive integers  $u_0$  and  $r$  and for  $0 \leq k \leq n$ , define  $u_k := u_0 + kr$ . Let  $L_n := \text{lcm}(u_0, u_1, \dots, u_n)$  and let  $a, l \geq 2$  be any integers. In this paper, we show that, for integers  $\alpha \geq a$  and  $r \geq \max(a, l-1)$  and  $n \geq l\alpha r$ , we have

$$L_n \geq u_0 r^{(l-1)\alpha + a - l} (r+1)^n.$$

Particularly, letting  $l = 3$  yields an improvement to the best previous lower bound on  $L_n$  obtained by Hong and Kominers.

## 1. INTRODUCTION

Hanson and Nair initiated the search for effective estimates for the least common multiple of the terms in a finite arithmetic progression; and, in [6] and in [13] they managed to produce good upper and lower bounds for  $\text{lcm}(1, 2, \dots, n)$ . In particular, Nair [13] discovered a nice new proof for the following well-known nontrivial lower bound

$$(1.1) \quad \text{lcm}(1, 2, \dots, n) \geq 2^{n-1}$$

for any integer  $n \geq 1$ . In [4], Farhi provided an identity involving the least common multiple of binomial coefficients and then use it to give a simple proof of the estimate (1.1). Inspired by Hanson's and Nair's works, Bateman, Kalb, and Stenger [1] and Farhi [2] respectively sought asymptotics and nontrivial lower bounds for the least common multiples of arithmetic progressions. Recently, Hong, Qian and Tan [10] extended the Bateman-Kalb-Stenger theorem from the linear polynomial to the product of linear polynomials. On the other hand, Farhi [2] obtained several nontrivial bounds and posed a conjecture which was later confirmed by Hong and Feng [7]. Hong and Feng [7] also got an improved lower bound for sufficiently long arithmetic progressions; this result was later sharpened further by Hong and Yang [11]. We notice that Hong and Yang [12] and Farhi and Kane [5] obtained some related results regarding the least common multiple of a finite number of consecutive integers. The theorem of Farhi and Kane [5] was extended by Hong and Qian [9] from the set of positive integers to the general arithmetic progression case. Recently, Qian, Tan and Hong [14] obtained some results about the least common multiple of consecutive terms in a quadratic progression.

In this paper, we study finite arithmetic progressions  $\{u_k := u_0 + kr\}_{k=0}^n$  with  $u_0, r \geq 1$  being integers satisfying  $(u_0, r) = 1$ . Throughout, we define  $L_n := \text{lcm}(u_0, u_1, \dots, u_n)$  to be the least common multiple of the sequence  $\{u_k\}_{k=0}^n$ . We begin with the following

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lower bound on  $L_n$ :

**Theorem 1.1.** [11] *Let  $\alpha \geq 1$  be an integer. If  $n > r^\alpha$ , then we have  $L_n \geq u_0 r^\alpha (r+1)^n$ .*

If  $r = 1$ , then Theorem 1.1 is the conjecture of Farhi [2] proven by Hong and Feng [7]. If  $\alpha = 1$ , then Theorem 1.1 becomes the improved lower bound of Hong and Feng [7]. In [8], Hong and Kominers sharpened the lower bound in Theorem 1.1 whenever  $\alpha, r \geq 2$ . In particular, they proved the following theorem which replaces the exponential condition  $n > r^\alpha$  of Theorem 1.1 with a linear condition  $n \geq 2\alpha r$ .

**Theorem 1.2.** [8] *Let  $a \geq 2$  be any given integer. Then for any integers  $\alpha, r \geq a$  and  $n \geq 2\alpha r$ , we have  $L_n \geq u_0 r^{\alpha+a-2} (r+1)^n$ .*

Letting  $a = 2$ , we see that Theorem 1.2 improves upon Theorem 1.1 for all but three choices of  $\alpha, r \geq 2$ . In the present paper, we provide a more general lower bound as follows.

**Theorem 1.3.** *Let  $a, l \geq 2$  be any given integers. Then for any integers  $\alpha \geq a$  and  $n \geq \max(a, l-1)$  and  $n \geq l\alpha r$ , we have  $L_n \geq u_0 r^{(l-1)\alpha+a-l} (r+1)^n$ .*

Picking  $l = 2$ , then Theorem 1.3 becomes Theorem 1.2. Letting  $l = 3$  in Theorem 1.3 gives us the following new lower bound.

**Theorem 1.4.** *Let  $a \geq 2$  be any given integer. Then for any integers  $\alpha, r \geq a$  and  $n \geq 3\alpha r$ , we have  $L_n \geq u_0 r^{2\alpha+a-3} (r+1)^n$ .*

Since  $\alpha \geq a \geq 2$ , we have  $2\alpha + a - 3 > \alpha + a - 2$ . Therefore the lower bound in Theorem 1.4 is better than that of Theorem 1.2 when  $n$  is large enough.

This paper is organized as follows. In Section 2, we first introduce relevant notation and previous results. Finally, we prove Theorem 1.3.

## 2. PROOF OF THEOREM 1.3

For any real numbers  $x$  and  $y$ , we say that  $y$  *divides*  $x$  if there exists an integer  $z$  such that  $x = yz$ . If  $x$  divides  $y$ , then we write  $y \mid x$ . As usual, we let  $\lfloor x \rfloor$  denote the largest integer no more than  $x$ .

Following Hong and Yang [11], we denote, for each integer  $0 \leq k \leq n$ ,

$$C_{n,k} := \frac{u_k \cdots u_n}{(n-k)!}, \quad L_{n,k} := \text{lcm}(u_k, \dots, u_n).$$

From the latter definition, we have that  $L_n = L_{n,0}$ .

The following Lemma first appeared in [2] and was reproved in [3] and [7].

**Lemma 2.1.** [2] [3] [7] *For any integer  $n \geq 1$ ,  $C_{n,0} \mid L_n$ .*

From Lemma 2.1, we see immediately that

$$(2.1) \quad L_{n,k} = A_{n,k} \frac{u_k \cdots u_n}{(n-k)!} = A_{n,k} \cdot C_{n,k}$$

for some integer  $A_{n,k} \geq 1$ .

Following Hong and Feng [7] and Hong and Yang [11], we define, for any  $n \geq 1$ ,

$$(2.2) \quad k_n := \max \left\{ 0, \left\lfloor \frac{n - u_0}{r + 1} \right\rfloor + 1 \right\}.$$

Hong and Feng [7] proved the following result.

**Lemma 2.2.** [7] *For all  $n \geq 1$  and  $0 \leq k \leq n$ ,*

$$L_n \geq L_{n,k_n} \geq C_{n,k_n} \geq u_0(r + 1)^n.$$

Now we are in a position to prove a lemma whose proof closely follows the approach of Hong and Yang [11].

**Lemma 2.3.** *Let  $a, l \geq 2$  be any given integers. Then for any integers  $\alpha \geq a$  and  $r \geq \max(a, l - 1)$  and  $n \geq l\alpha r$ , we have  $n - k_n > ((l - 1)\alpha + a - l)r$ .*

*Proof.* If  $n \leq u_0$ , then by the definition (2.2),  $k_n \leq 1$ . Since  $\alpha, r \geq a \geq 2$  and  $n \geq l\alpha r$ , we derive that  $n - k_n \geq n - 1 \geq l\alpha r - 1 > ((l - 1)\alpha + a - l)r$ .

Now we suppose that  $n > u_0$ . In this case, we have

$$k_n = \left\lfloor \frac{n - u_0}{r + 1} \right\rfloor + 1.$$

So we have

$$k_n \leq \frac{n - u_0}{r + 1} + 1 \leq \frac{n - 1}{r + 1} + 1 = \frac{n + r}{r + 1}.$$

It then follows that

$$(2.3) \quad n - k_n \geq n - \frac{n + r}{r + 1} = \frac{(n - 1)r}{r + 1} \geq \frac{(l\alpha r - 1)r}{r + 1}.$$

Note that  $r \geq l - 1$  tells us that  $r - l + 1 \geq 0$ . Then from the assumption  $\alpha, r \geq a$  it follows that

$$(2.4) \quad \begin{aligned} (l\alpha r - 1) - (r + 1)((l - 1)\alpha + a - l) &= (r - l + 1)\alpha - 1 - (r + 1)(a - l) \\ &\geq a(r - l + 1) - 1 - (r + 1)(a - l) \\ &= l(r - a) + l - 1 > 0. \end{aligned}$$

Therefore by (2.4), we infer that

$$(2.5) \quad \frac{l\alpha r - 1}{r + 1} > (l - 1)\alpha + a - l.$$

The desired result then follows immediately from (2.3) and (2.5).  $\square$

Using the similar argument as that of Theorem 1.1, by Lemma 2.3 we can now prove Theorem 1.3 as the conclusion of this paper.

*Proof of Theorem 1.3.* By hypothesis, we have  $\alpha, r \geq a \geq 2$ ,  $l \geq 2$  and  $n \geq l\alpha r$ . It follows from Lemma 2.3 that  $r^{(l-1)\alpha+a-l} \mid (n - k_n)!$ . Thus, we may express  $(n - k_n)!$  in the form  $r^{(l-1)\alpha+a-l} \cdot B_n = (n - k_n)!$ , with  $B_n \geq 1$  being an integer. Letting  $k = k_n$  in (2.1), we find that

$$r^{(l-1)\alpha+a-l} \cdot B_n \cdot L_{n,k_n} = A_{n,k_n} \cdot u_{k_n} \cdots u_n.$$

It then follows that  $r^{(l-1)\alpha+a-l} \mid A_{n,k_n}$ , since the requirement  $(r, u_0) = 1$  implies that  $(r, u_k) = 1$  for all  $0 \leq k \leq n$ . Then, we get from (2.1) and Lemma 2.2 that

$$L_{n,k_n} \geq r^{(l-1)\alpha+a-l} C_{n,k_n} \geq u_0 r^{(l-1)\alpha+a-l} (r+1)^n.$$

Therefore the statement of Theorem 1.3 follows immediately. The proof of Theorem 1.3 is complete.  $\square$

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